International Journal of Theoretical Physics, Vol. 3, No. 6 (1970), pp. 483–505

Observer Theories Based on Stueckelberg Equations of Motion

ROBERT G. CAWLEY

U.S. Naval Ordnance Laboratory, Silver Spring, Maryland 20910

Received: 17 *June* 1970

Abstract

Stueckelberg dynamics is regarded as providing a basis for the construction of observer centered theories of particle motions. The approach involves the use of a generalized Jacobi principle to replace the four-dimensional dynamical theory of Stueckelberg by a four-dimensional geometrical theory, and then a three-dimensional dynamics is constructed from this. The causal difficulties encountered by Stueckelberg for curves which reverse direction in time appear to be absent in the present scheme.

Our purpose has been to make more concrete, in a simple context, some of the ideas involved in the (conventional) causal framework recently constructed by us to deal with causal difficulties associated with hyperlight phenomena. Some insight is gained into the possible roles to be played by tachyons in a particle theory and interesting results are found involving classical Lagrangian and canonical formalisms for lightlike particles.

1. *Introduction*

The motion of a system of N point particles obeying classical equations derivable from an Euler–Lagrange principle of stationary action is described by the behavior of the system point in a $3N$-dimensional configuration space. Here the time parametrizes the motion, but it is also possible to represent the system by a point in a $(3N+1)$-dimensional extended space by treating the time as a coordinate. In the extended space, only the system point trajectory bears a unique relation to the configuration space motion, and corresponding to this the extended space Lagrangian is a homogeneous function of the first degree in the velocities and the evolution parameter is arbitrary.†

In the customary theoretical approach to the problems of particle mechanics physical laws are expressed with the aid of dynamical equations which depict the motion of the particles in time. This corresponds to the use of the configuration space to represent the behavior of a system as determined by the given laws. This is a specifically observed centered

† See for example Lanczos, C. P. (1957). *The Variational Principles of Mechanics*. University of Toronto Press, Toronto, Ontario, Canada.

approach, as observers formulate laws whose cause–effect relation generates their (forward) 'motion' in time. On the other hand, there is no physical motion in the extended space but something rather more like an overall space and time plot from which configuration space motions can be determined. A theoretical approach which starts from extended space considerations is the following: from regularities exhibited by allowed classes of extended space trajectories construct an observer centered representation in terms of laws governing a configuration space time evolution.

We may suppose that in the first of these approaches experiment suggests the laws, while in the second it suggests the allowed classes of extended space trajectories. The sharp separation of space and time found in the kinematics of nonrelativistic mechanics and in the configuration space formalism makes the first approach natural in nonrelativistic theories. In relativistic theories the second approach has the advantage that relativistic invariance requirements are readily expressed by means of four-dimensional geometry.

Our purpose is to explore some of the possibilities of the second of these approaches in theories of relativistic particle mechanics. To generate classes of extended space trajectories we employ a technique based on the use of a Stueckelberg action principle (Stueckelberg, 1941, 1942). Our method parallels that used in the derivation of Jacobi's principle and it leads to variation principles for Minkowski trajectories.

There is a subtle but important shift in viewpoint from that of Stueckelberg in the present approach; its spirit is that of the (conventional) causal framework recently constructed by the author to deal with causality problems associated with hyperlight phenomena (Cawley, 1970). We regard the observer's role as that of interpreter and classifier of the regularities of the patterns exhibited by the Minkowski trajectories. The observer then describes these in terms of 'laws which govern the behavior of systems'. An example of the way this shifts some notions of physical 'reality' from the world observed to the observer is provided in the remark that particles do not 'go' in either direction in time; the time sense of which we are aware is macroscopic and is a consequence of the observer's mode of representation of natural phenomena resulting from the cause-effect relation of physical laws. Indeed, the structure of space and time itself reflects physical properties of the observer.† Though the regularities of the curves are determined by a Stueckelberg action principle which gives equations of motion for all *four* coordinates, our theory *is not* an example of 'four-dimensional dynamics', because the observer representation singles out the time coordinate in every frame for use as a parameter in the formulation of laws. This is partly formalized by our deriving action principles for the *trajectories* from the primary Stueckelberg action principle for the four-dimensional 'motions'.

† The assertion that this statement is false is unprovable because space and time are (at least!) *observer constructs.*

In Section 2 we treat the general theory of the Stueckelberg action and the Stueckelberg–Jacobi formalism, applying it to free particle examples and particles in external fields. In Section 3 we discuss theories in which the lengths of the world line tangent vectors are not constant. In Section 4 we consider canonical theories and in Section 5 we make a few concluding remarks.

2. *Independent Trajectories Theory*

A. *General Theory* (*Single Trajectory Case*)

We consider the class of continuous, twice-differentiable curves in Minkowski space,

$$\Gamma:\lambda \to x^{\mu} = x^{\mu}(\lambda), \qquad \lambda_1 < \lambda < \lambda_2 \qquad \text{and} \qquad \mu = 0 \text{ to } 3 \tag{2.1}$$

having the property that the functions $x^{\mu}(\lambda)$ are derivable as solutions to Euler–Lagrange equations resulting from a variation principle based on the Stueckelberg action

$$A_S = \int_{\lambda_1}^{\lambda_2} d\lambda L_S(x(\lambda), dx(\lambda)/d\lambda) \tag{2.2}$$

The Stueckelberg Lagrangian L_S is required to have no explicit λ-dependence. The Euler–Lagrange equations of the variation principle defined by the requirement (to first order is always understood),

$$\delta A_S = 0 \tag{2.3}$$

with the variation of A_S generated by independent infinitesimal variations of the $x^{\mu}(\lambda)$ satisfying $\delta x^{\mu}(\lambda_1) = \delta x^{\mu}(\lambda_2) = 0$ and with $\delta(dx^{\mu}/d\lambda) = (d/d\lambda) \times \delta x^{\mu}(x)$ possess the integral†

$$R(x, dx/d\lambda) = (dx^{\mu}/d\lambda)\,\frac{\partial L_S}{\partial(dx^{\mu}/d\lambda)} - L_S = -p_{\lambda} = \text{constant} \tag{2.4}$$

This follows from the λ-independence of $L_S(x, dx/d\lambda)$.

We suppose that x^{μ} and λ can be expressed as functions of a new variable α by

$$x^{\mu} = x^{\mu}(\alpha) \qquad \text{and} \qquad \lambda = \lambda(\alpha) \tag{2.5}$$

so that A_S becomes

$$A_S = \int_{\alpha_1}^{\alpha_2} d\alpha\lambda' L_S\ (x, x'/\lambda') = \int_{\alpha_1}^{\alpha_2} d\alpha \bar{L}_S(x, x', \lambda') \tag{2.6}$$

where $\lambda(\alpha_1) = \lambda_1$ and $\lambda(\alpha_2) = \lambda_2$ and primes denote differentiation by α. In equation (2.5) we require that $\lambda'(\alpha)$ be nonvanishing as this entails no

† We use the Einstein summation convention for repeated upper and lower indices and the space-favoring metric for Minkowski space ($g^{11} = g^{22} = g^{33} = -g^{00} = +1$).

loss of generality. If we denote by $\delta' A_S$ the variation of the last member of equation (2.6) generated by arbitrary independent fixed endpoint variations of $x^\mu(\alpha)$ and $\lambda(\alpha)$, with $[\delta', d/d\alpha] = 0$ giving $\delta'(dx^\mu/d\alpha)$ and $\delta'(d\lambda/\partial\alpha)$, then equation (2.3) implies

$$\delta' A_S = 0 \tag{2.7}$$

under the stated conditions. Conversely, using $\lambda'(\alpha) \neq 0$, one finds that the Euler–Lagrange equations generated by the variation principle based equation (2.7) include those following from (2.3) together with an additional equation equivalent to (2.4). Hence (2.7) can replace (2.3) and λ may be regarded as a coordinate.

Since L_S has no explicit λ-dependence, λ is cyclic and we can eliminate it from $\bar{L}_S$ by solving

$$F(x, x', \lambda') = p_\lambda - \frac{\partial \bar{L}_S(x, x', \lambda')}{\partial \lambda'} = 0 \tag{2.8}$$

for λ' and substituting, provided a solution exists. To complete the elimination of λ we must also reformulate the action principle in terms of variations of the $x^\mu(\alpha)$ alone. Let $\delta'' A_S$ denote the variation of A_S so generated, with equation (2.8) regarded as having been solved for λ' and the solution as substituted into equation (2.6). Since equation (2.7) holds for *arbitrary* fixed endpoint infinitesimal variations of x^μ and λ, and therefore for those restricted by equation (2.8), then, regarding x^μ and λ as intermediate variables, the only contribution to $\delta'' A_S$ is from the endpoint terms for the λ-part; there could be such contributions as it is possible that equation (2.8) might not allow $\delta''\lambda(\alpha_1) = \delta''\lambda(\alpha_2) = 0$. Using equation (2.7) we have

$$\delta'' A_S = \frac{\partial \bar{L}_S(x, x', \lambda')}{\partial \lambda'} \delta''\lambda\big|_{\alpha_1}^{\alpha_2} = p_\lambda \delta''\lambda\big|_{\alpha_1}^{\alpha_2} \tag{2.9}$$

and by the required constancy of p_λ,

$$\delta'' A_S = \delta \int_{\alpha_1}^{\alpha_2} d\alpha \bar{L}_S = p_\lambda \int_{\alpha_1}^{\alpha_2} d\alpha \frac{d(\delta''\lambda)}{d\alpha}$$

$$= \delta'' \int_{\alpha_1}^{\alpha_2} d\alpha \lambda' p_\lambda \tag{2.10}$$

whereupon

$$\delta'' A_{SJ} = \delta'' \int_{\alpha_1}^{\alpha_2} d\alpha L_{SJ}(x, x') = 0 \tag{2.11}$$

in which

$$L_{SJ}(x, x') = \bar{L}_S(x, x', \lambda') - \lambda' p_\lambda \tag{2.12}$$

with λ' given implicitly by equation (2.8). It is important to note that while α is arbitrary *some* parameter must be chosen in (2.11); the new configuration space has only *three* independent degrees of freedom, *not four*.

We can rewrite equation (2.8) by substituting

$$\bar{L}_S(x, x', \lambda') = \lambda' L_S(x, x'/\lambda') \tag{2.13}$$

and with the help of equation (2.4) find

$$F(x, x', \lambda') = p_\lambda + R(x, x'/\lambda') = 0 \tag{2.14}$$

Substituting this result into equation (2.12) gives

$$L_{SJ}(x, x') = (L_S(x, x'/\lambda') + R(x, x'/\lambda')) \tag{2.15}$$

$$= x' . \frac{\partial L_S(x, x'/\lambda')}{\partial(x'/\lambda')} \tag{2.16}$$

We observe that equation (2.16) reveals $L_{SJ}(x, x')$ to be a homogeneous function of the first degree in the velocities $x'(\alpha)$.

It may sometimes happen that equation (2.8), or equivalently equation (2.14), is satisfied for arbitrary λ'; in this case λ' cannot be eliminated from L_{SJ} and it appears in equation (2.15) as an undetermined function of α. As arbitrary infinitesimal variations of λ are allowed in equation (2.11) so are arbitrary variations of

$$\mu = M(\lambda') \tag{2.17}$$

where M is any function. In particular, fixed endpoint variations of μ produce no additional contribution to $\delta'' A_{SJ}$ and we may regard μ as an extra coordinate if we so choose. This makes L_{SJ} a function of μ as well as x and x',

$$L_{SJ} = L_{SJ}(x, \mu, x') \tag{2.18}$$

and results in an additional Euler–Lagrange equation

$$\frac{\partial L_{SJ}(x, \mu, x')}{\partial \mu} = 0 \tag{2.19}$$

We observe that L_{SJ} no longer has to be a homogeneous function of the first degree in the velocities, but the coordinates number four, not five.

To determine the content of equation (2.19) we note that

$$\frac{\partial L_{SJ}(x, \mu, x')}{\partial \mu} = -\left(\frac{\lambda'^2\, dM(\lambda')}{d\lambda'}\right)^{-1} \frac{\partial L_{SJ}(x, 1/\lambda', x')}{\partial(1/\lambda')}$$

and by equation (2.15)

$$\frac{\partial L_{SJ}(x, 1/\lambda', x')}{\partial(1/\lambda')} = -\lambda'^2 (L_S(x, x'/\lambda') + R(x, x'/\lambda')) + \lambda' x' . \left(\frac{\partial L_S(x, x'/\lambda')}{\partial(x'/\lambda')} + \frac{\partial R(x, x'/\lambda')}{\partial(x'/\lambda')}\right)$$

$$= -\lambda' L_{SJ} + \lambda' L_{SJ} + \lambda' x' . \frac{\partial R(x, x'/\lambda')}{\partial(x'/\lambda')} \tag{2.20}$$

where we used equation (2.16) in the last line. Hence, as λ' is not permitted to vanish, equation (2.19) is equivalent to

$$x' . \frac{\partial R(x, x'/\lambda')}{\partial(x'/\lambda')} = 0 \tag{2.21}$$

But regarding R as a function of x, $1/\lambda'$ and x' we note that

$$\frac{\partial R}{\partial(1/\lambda')} = \frac{\partial(x'/\lambda')}{\partial(1/\lambda')} . \frac{\partial R(x, x'/\lambda')}{\partial(x'/\lambda')}$$

$$= x' . \frac{\partial R(x, x'/\lambda')}{\partial(x'/\lambda')} \tag{2.22}$$

so equation (2.21) becomes

$$\frac{\partial R(x, 1/\lambda', x')}{\partial(1/\lambda')} = 0 \tag{2.23}$$

whence by equation (2.14)

$$\frac{\partial F(x, x', \lambda')}{\partial \lambda'} = 0 \tag{2.24}$$

This result can be understood as follows. We return momentarily to the formulation of (2.11) which regards just the x^μ as coordinates and in which λ' appears in equation (2.15) and in the equations of motion for the x^μ as an undetermined function of α. That equation (2.14) is satisfied for any λ' means that arbitrary variations of λ' produce no change in F. This fact has two consequences, the first being the equivalence of equations (2.14) and (2.24) and the second is that equation (2.14) must be expressible as

$$F(x, x') = 0 \tag{2.25}$$

Equation (2.25) now selects from the solutions to the x^μ-equations a *particular* first integral. Here (the α-dependence of) $\lambda'(\alpha)$ will be determined from the equations of motion for the x^μ deriving from (2.11) by means of the requirement that equation (2.25) be satisfied.

Thus there are two equivalent formulations of the variation principle (2.11) each giving the same Euler–Lagrange equations for the x^μ. In one of these the Lagrangian and the equations of motion contain an arbitrary function of α and equation (2.25) appears as a subsidiary condition to these equations. In the other the presence of an arbitrary function gives way to the appearance of an additional coordinate whose equation of motion reduces to equation (2.25). The second of these is more convenient for passage to a canonical formalism.

Finally it may be that in equation (2.14) R vanishes identically; this holds in the 'uninteresting' case that L_S is homogeneous of the first degree in $dx/d\lambda$. In that event the variation principle (2.3) already gives trajectories and λ may be assigned the role of the arbitrary parameter α from the start.

We note that the addition to L_S of a term $(dx/d\lambda).\partial \Lambda(x) = d\Lambda(x(\lambda))/d\lambda$ in equation (2.2) has no effect on the variation of the Stueckelberg action. This shows up in $L_{SJ}(x, x')$ as the addition of the term

$$x'(\alpha).\partial\Lambda(x) = \frac{d\Lambda(x(\alpha))}{d\alpha}$$

and is therefore unobservable. Finally the variable change, $\lambda \to \bar{\lambda} = a\lambda$, leaves the formalism invariant at the Stueckelberg level because this still leaves L_S with no explicit λ-dependence. This transformation is discussed later (cf. Section 2, D).

B. *Single Trajectory Examples* (*No External Fields*)

We consider the Stueckelberg Lagrangian,

$$L_S = \frac{1}{2}\left(\frac{dx}{d\lambda}\right)^2 \tag{2.26}$$

and from equations (2.4) and (2.14) we have

$$p_\lambda = -R = -\frac{1}{2}\left(\frac{dx}{d\lambda}\right)^2 = -\frac{\frac{1}{2}x'^2}{\lambda'^2} \tag{2.27}$$

so the signature of the tangent vector, i.e. the sign of $(dx/d\lambda)^2$, either +, –, or 0, is constant. If $p_\lambda \neq 0$ we have

$$\lambda'(\alpha) = \pm\left[-\frac{x'(\alpha)^2}{2p_\lambda}\right]^{1/2} \tag{2.28}$$

which must be real, and

$$L_{SJ}(x, x') = +\lambda'.\frac{x'^2}{\lambda'^2} = \pm m\epsilon((x')^2)\,|(x')^2|^{1/2} \tag{2.29}$$

wherein $|2p_\lambda|^{1/2} > 0$ has been denoted by m. If $p_\lambda = 0$, equation (2.27) does not determine λ', but instead reduces to the equation

$$x'(\alpha)^2 = 0 \tag{2.30}$$

in addition the Lagrangian (2.29) now has explicit α-dependence,

$$L_{SJ}(x, x', \alpha) = (\lambda'(\alpha))^{-1}(x')^2 \tag{2.31}$$

with $\lambda'(\alpha)$ an arbitrary function of α. This arbitrariness and the subsidiary condition (2.30) may be traded for a new coordinate

$$\mu = \frac{2}{\lambda'(\alpha)} \tag{2.32}$$

giving

$$L_{SJ}(x, x', \mu) = \tfrac{1}{2}\mu . (x')^2 \tag{2.33}$$

For the bradyon (timelike) case $(x')^2 < 0$, and if we identify α with the time by choosing $\alpha = x^0 = t$ we get

$$L_{SJ} = \mp m_b(1 - \dot{\mathbf{x}}^2)^{1/2} \tag{2.34}$$

where $\dot{\mathbf{x}} = d\mathbf{x}/dx^0 = d\mathbf{x}/dt$. The double sign comes from (2.28) so that in the one case (upper sign) λ increases with the time and in the other case it decreases.

The Stueckelberg–Jacobi action integral runs over increasing values of λ and this is an inappropriate base from which to formulate an observer representation. Instead we base the theoretical structure on the invariant observer time sense by introducing the observer action as an integral over increasing time,

$$A_\Omega = \int_{t_1}^{t_2} dt L_\Omega(\mathbf{x}, \dot{\mathbf{x}}) \tag{2.35}$$

where the limits are defined by

$$t_1 = \min(x^0(\lambda_1), x^0(\lambda_2)) < t_2 = \max(x^0(\lambda_1), x^0(\lambda_2)) \tag{2.36}$$

and the observer Lagrangian is

$$L_\Omega = -m_b(1 - \dot{\mathbf{x}}^2)^{1/2} \tag{2.37}$$

When $x^\mu(\lambda_1) - x^\mu(\lambda_2)$ is timelike, as it must be in the present example, the relation $t_1 < t_2$ in (2.36) is Lorentz invariant. We may regard one of the endpoint events $x^\mu(\lambda_i)$ as a production event (cause) and the other as a detection event (effect). Note that the observer representation does not distinguish the two possibilities $\dot{\lambda}(t) > 0$ and $\dot{\lambda}(t) < 0$.

For the tachyon (spacelike) case we may also choose to identify α with the time coordinate, getting

$$L_{SJ} = \pm m_t(\dot{\mathbf{x}}^2 - 1)^{1/2} \tag{2.38}$$

For hyperlight motion, however, the vector $x(\lambda_1) - x(\lambda_2)$ can be spacelike, and in that case it is not possible to write the action as an integral over increasing time in an invariant way. Corresponding to this a tachyon curve has an eventlike character rather than a particlelike character (Cawley, 1970), with the equations of 'motion' more properly regarded as determining the shape of the line. Consequently, in contrast to the bradyon case, the observer representation does not require the introduction of a special observer action integral.

We can illustrate these remarks by working out the free tachyon example. Suppose coordinates be chosen so that the x^3-axis is along the space part of the vector $x(\lambda_1) - x(\lambda_2)$. Choosing $\alpha = x^3 = z$ gives

$$L_{SJ} = \pm m_t(1 + (\dot{x}^1)^2 + (\dot{x}^2)^2 - (\dot{x}^0)^2)^{1/2} \tag{2.39}$$

where the dot now means differentiation by z. The Euler–Lagrange equations based on equations (2.11) and (2.39), with the boundary conditions,

$$x^1(0) = x^2(0) = x^0(0) = x^1(L) = x^2(L) = 0, \qquad x^0(L) = T \tag{2.40}$$

have the solution,

$$x^1(z) = x^2(z) = 0, \qquad x^0(z) = \frac{T}{L}z \tag{2.41}$$

We regard the endpoint events as elementary detection-production events and the effect of the absence of external fields over the region spanned by the tachyon line as its straight line shape. Because the world line is spacelike neither one of these events can be regarded as *the* production event (cause) in a physically meaningful way.† Note that the observer representation does not distinguish between the two possible signs of $\dot{\lambda}(z)$ because the causal labelling of the endpoint events is symmetric.

For the photon (lightlike‡) case, again choosing $\alpha = x^0 = t$ so that equation (2.32) reads

$$\mu = \frac{2}{\dot{\lambda}(t)} \tag{2.42}$$

we have

$$L_{SJ}(\mathbf{x}, \mu, \dot{\mathbf{x}}) = \tfrac{1}{2}\mu(\dot{\mathbf{x}}^2 - 1) \tag{2.43}$$

Passing to the observer representation we have

$$A_\Omega = \int_{t_1}^{t_2} dt L_\Omega(\mathbf{x}, \omega, \dot{\mathbf{x}}) = \int_{t_1}^{t_2} dt \tfrac{1}{2}\omega(\dot{\mathbf{x}}^2 - 1) \tag{2.44}$$

where t_1 and t_2 are given by (2.36) and where

$$\omega = |\mu| > 0 \tag{2.45}$$

† We have discussed in detail in Ref. 3 why in the conventional causal framework a tachyon must be regarded as a spatially extended line of matter possessing an eventlike character rather than as an elementary particle. Thus it is the entire event complex constituted in the whole world line whose production is controlled by the observer rather than one of these events (the earliest) as would be the case for a genuine particle. Despite this fundamental observational property of tachyons it is sometimes useful to speak of them as if they were particles (extended causal framework) and we often do this for economy of presentation.

‡ We use the word photon in the generic sense to include any particle which moves with the invariant speed.

That ω cannot vanish follows from the fact that the Euler–Lagrange equations of the Stueckelberg action principle,

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \tag{2.46}$$

do not possess a nontrivial solution with $dx^0/d\lambda = 0$ and $(dx^\mu/d\lambda)^2 = 0$.

The observer equations of motion are

$$\frac{d(\omega\dot{\mathbf{x}})}{dt} = 0, \qquad \dot{\mathbf{x}}^2 - 1 = 0 \tag{2.47}$$

which gives

$$\dot{\omega} = 0, \qquad \omega\dot{\mathbf{x}} = \mathbf{k} \neq 0 \tag{2.48}$$

whence

$$\mathbf{x}(t) = \hat{k}t + \mathbf{x}(0), \qquad \hat{k}^2 \equiv 1 \tag{2.49a}$$

$$\omega = |\mathbf{k}| \tag{2.49b}$$

where $\mathbf{k}$ is an arbitrary nonzero constant three-vector. We note that again the observer representation does not distinguish between the two possibilities $\dot{\lambda}(t) > 0$ and $\dot{\lambda}(t) < 0$.

C. *Single Trajectory Examples* (*External Fields*)

In a mechanical theory of particles we may be able to approximate the effect of a large number of particles on a single particle by means of a representation involving what observers can call an external field. This may in fact be the case for classical electrodynamics of particles† and in any event it provides a convenient starting point for further investigation of the Stueckelberg-Jacobi formalism.

We consider the class of Stueckelberg Lagrangians,

$$L_S(x, dx/d\lambda) = \sum_{n=0} L_S^{(n)}(x, dx/d\lambda) \tag{2.50}$$

where $L_S^{(n)}$ is a homogeneous function of $dx/d\lambda$ of degree n. By Euler's theorem on homogeneous functions,

$$\frac{(dx^\mu/d\lambda)\,\partial L_S^{(n)}}{\partial(dx^\mu/d\lambda)} = nL_S^{(n)} \tag{2.51}$$

so

$$R = -L_S^{(0)} + L_S^{(2)} + 2L_S^{(3)} + \cdots \tag{2.52}$$

and

$$L_{SJ} = x' \cdot \frac{\partial L_S(x, x'/\lambda')}{\partial(x'/\lambda')} = \lambda' \sum_{n=0} nL_S^{(n)} \tag{2.53}$$

† Although the self-energy problem for charged tachyons seems to be more serious than usual (Cawley, 1970).

Equation (2.52) shows that for examples with $L_S^{(n)} = 0$, for $n \neq 1$ or 2, and $L_S^{(2)} = \frac{1}{2}(dx/d\lambda)^2$, the constancy of R guarantees that of the signature of the tangent vector $x'(\alpha)$. In such cases the equations of the observer representation are such as to forbid motions involving accelerations of particles in such a way that they change their speed signature from bradyon to tachyon or vice versa. Also, timelike trajectories have $\dot{\lambda}(t) > 0$ or $\dot{\lambda}(t) < 0$ and passage from one classification to the other is impossible. For such examples as this, and for $p_\lambda \neq 0$, we find from equations (2.27)–(2.29) and (2.53),

$$L_{SJ}(x, x') = \pm m\epsilon((x')^2)\,|(x')^2|^{1/2} + x' \,.\, C(x, x') \tag{2.54}$$

where

$$C(x, x') = \left.\frac{\partial L_S^{(1)}(x, dx/d\lambda)}{\partial(dx/d\lambda)}\right|_{dx/d\lambda = x'/\lambda'} \tag{2.55}$$

$C(x, x')$ does not depend on the sign of $\lambda'(\alpha)$ because the indicated derivative of $L_S^{(1)}$ is a homogeneous function of degree zero in $dx/d\lambda$. For $p_\lambda = 0$ we have

$$L_{SJ}(x, \mu, x') = \tfrac{1}{2}\mu \,.\, (x')^2 + x' \,.\, C(x, x') \tag{2.56}$$

Some examples of coupling to external fields by means of first degree Stueckelberg Lagrangians are

$$L_S^{(1)} = f\left|\left(\frac{dx}{d\lambda}\right)^2\right|^{1/2} \Phi(x) \qquad \text{(scalar field)} \tag{2.57a}$$

$$L_S^{(1)} = q\frac{dx}{d\lambda} \cdot A(x) \qquad \text{(vector field)} \tag{2.57b}$$

$$L_S^{(1)} = \gamma\left|\left(\frac{dx}{d\lambda}\right)^2\right|^{-1/2} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} G_{\mu\nu}(x) \qquad \text{(tensor field)} \tag{2.57c}$$

The second of these is the most familiar, and for the bradyon case it generates the observer Lagrangians

$$L_\Omega^{(\pm)}(\mathbf{x}, \dot{\mathbf{x}}, t) = -m_b(1 - \dot{\mathbf{x}}^2)^{1/2} \pm q(\dot{\mathbf{x}} \,.\, \mathbf{A} - A^0) \tag{2.58}$$

which follows from (2.35), (2.36), (2.54) and (2.55). Since the trajectories are everywhere timelike, the relation $t_1 < t_2$ of (2.36) is again an invariant one. We see that here the observer representation distinguishes the cases $\dot{\lambda}(t) > 0$ and $\dot{\lambda}(t) < 0$ for a given L_S. The scalar and tensor examples can be treated in the same way and with similar results.

For the tensor field coupling (2.57c) to photon trajectories, L_S does not exist anywhere on the actual paths, and for the scalar example $\partial L_S/\partial(dx/d\lambda)$ does not exist. In the first of these the expression (2.2) is meaningless, and in the second there is difficulty in evaluating the left side of equation (2.3) unless the variations are restricted to $(dx/d\lambda)^2 = 0$. But if this is done then (2.3) allows *arbitrary* motion satisfying this condition. To avoid these problems in the photon case we confine our remarks to the vector field example.

First we show that the Minkowski trajectories cannot reverse direction in the time coordinate. The condition for reversal to occur is that $dx^0/d\lambda = \frac{1}{2}\mu$ pass through zero. At such a point the $dx^\mu/d\lambda$ must all vanish because of the defining condition $(dx/d\lambda)^2 = 0$. The Euler–Lagrange equations of the Stueckelberg action principle,

$$\frac{d^2 x^\mu}{d\lambda^2} = \frac{dx_\nu}{d\lambda}(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2.59}$$

show that all derivatives of x^μ must vanish there. But a curve, all of whose derivatives vanish at a point, is the physically uninteresting 'constant curve' into a single point of Minkowski space.

This means that not only does μ (invariantly) fail to vanish for all but the trivial case, but also it cannot change sign in the motion. Hence the observer Lagrangians for the charged lightlike particle are

$$L_\Omega^{(\pm)}(\mathbf{x}, \omega, \dot{\mathbf{x}}, t) = \tfrac{1}{2}\omega(\dot{\mathbf{x}}^2 - 1) \pm q(\dot{\mathbf{x}}.\mathbf{A}(x, t) - A^0(\mathbf{x}, t)) \tag{2.60}$$

where ω is given by (2.45). The observer representation again distinguishes the two kinds of trajectory by the sign of the coupling. Notice also that the invariance of (2.36), i.e. of the relation $t_1 < t_2$, is guaranteed by the invariant constancy of the sign of μ.

To discuss the tachyon case we recall first that for free bradyons or photons initial position and velocity can be controlled, thereby supplying the cause to the observed effect of final position and velocity, but that this cannot be done for tachyons because production at a given event is not physically meaningful.† This feature must govern the interaction of observer and tachyon even if the endpoint events of the Stueckelberg action integral are timelike or null separated, as they can be for the tachyon in an external field. This means that just as in the free tachyon case there is in general no observer action integral to be distinguished from the Stueckelberg–Jacobi action integral. The observer representation [for the Lagrangians involving equations (2.57)], does distinguish between the two classes of trajectory characterized by the two possible signs of $\lambda'(\alpha)$ for a given α, however, because there are two corresponding Stueckelberg–Jacobi Lagrangians, which give two different trajectories between a given event pair.

D. *Effect of Invariances of Stueckelberg Action*

The simplest invariance is Lorentz invariance, and this appears in the observer representation through the invariance of the functional form of the observer (or Stueckelberg–Jacobi) Lagrangian.

More interesting is the invariance to linear transformations of λ. While addition of a constant to λ has no effect in the observer Lagrangian and so is uninteresting, multiplication of λ by a nonzero constant induces a transformation of equation (2.4) which does survive. In the cases so far con-

† See footnote on p. 491. The remark of the text is intended to refer only to the context of the *conventional* causal framework.

sidered this results in a scale change of m [ω, if $p_\lambda = 0$; cf. equations (2.42) and (2.45)] and this can be absorbed, at the observer level, into a redefinition of the coupling strength. A sign change of m or ω is not possible because of the way these are defined. Transformations such as these are harmless, since the observer does not directly detect the four-dimensional dynamics of the Stueckelberg equations but merely represents in space and time language the regularities of the corresponding Minkowski trajectories. As we have already emphasized, this feature is the most important point of departure of the present approach from the original Stueckelberg theory.

3. *Examples with Variable* $(dx/d\lambda)^2$

If the coupling Lagrangian is not of the first degree in $(dx/d\lambda)$ the constancy of R does not result in world line tangent vectors of constant signature. Particles which experience bradyon motion in field-free regions can be accelerated to the speed of light and beyond,† and Minkowski trajectories can reverse direction (invariantly) in the time coordinate.

First we consider the example of Stueckelberg (1941, 1942),

$$L_S = \frac{1}{2}\left(\frac{dx}{d\lambda}\right)^2 - a\phi(x) \tag{3.1}$$

where $\phi(x)$ is a Lorentz scalar field. Equations (2.4) and (2.52) give

$$p_\lambda + \frac{1}{2}\left(\frac{dx}{d\lambda}\right)^2 = -a\phi(x) \tag{3.2}$$

so that

$$\lambda' = \pm\left[\frac{(x')^2}{-2p_\lambda - 2a\phi(x)}\right]^{1/2} \tag{3.3}$$

which must be real, and

$$L_{SJ} = \frac{(x')^2}{\lambda'} = \pm\epsilon((x')^2)\,[(x')^2(-2p_\lambda - 2a\phi(x))]^{1/2} \tag{3.4}$$

For simplicity of presentation we suppose for a moment that $\phi(x) = 0$ unless x belongs to a finite domain $\mathscr{D}$ of Minkowski space. By equation (3.2),

$$R = R_{\text{free}} = \frac{(x')^2}{2(\lambda')^2} = -p_\lambda, \qquad x \in \mathscr{D}' \tag{3.5}$$

where $\mathscr{D}'$ is the complement of $\mathscr{D}$. Observe that the mass parameter, defined by $m = |2p_\lambda|^{1/2}$, and the signature of $(dx/d\lambda)^2$ are the same for every segment of the trajectory intercepted by $\mathscr{D}'$.

† See footnote on p. 491.

We observe that equation (3.2) permits

$$\left(\frac{dx}{d\lambda}\right)^2 = 0 \tag{3.6a}$$

at events $\bar{x}$ of the trajectory, where

$$p_\lambda = -a\phi(\bar{x}) \tag{3.6b}$$

If the trajectory does not terminate in $\mathscr{D}$ and if it passes through $\mathscr{D}'$, then as we move along it from events in $\mathscr{D}'$ to events in $\mathscr{D}$ and back to events in $\mathscr{D}'$ the left side of equation (3.2) passes from zero through nonzero values *and back* to zero. If in $\mathscr{D}$, therefore, there is an event $\bar{x}_1$ at which equations (3.6) are satisfied, then there is another event $\bar{x}_2$ at which these equations are again satisfied, excepting the degenerate case that $\bar{x}_1 = \bar{x}_2$. The points $\bar{x}_1$ and $\bar{x}_2$ represent the events $\bar{e}_1$ and $\bar{e}_2$ at which the particle passes through the light 'barrier' exchanging bradyon motion for tachyon motion (or vice versa) and then passes back. When $p_\lambda = 0$ there are again two 'distinguished' events, the intersections of the trajectory with the boundary of $\mathscr{D}$, where the motion changes from lightlike to timelike or to spacelike.

Equations (3.6) induce an invariant breakup of the Minkowski trajectory into segments whose tangent vectors have constant signature and this results in a decomposition of the Stueckelberg–Jacobi action,

$$A_{SJ} = A_{SJ}[1\,\bar{1}] + A_{SJ}[\bar{2}\,2] + A_{SJ}[\bar{1}\,\bar{2}] \tag{3.7a}$$

where the partial actions are

$$A_{SJ}[rs] = \int_{\alpha_r}^{\alpha_s} d\alpha L_{SJ} \tag{3.7b}$$

When additional event pairs satisfy equations (3.6) corresponding terms appear in equation (3.7a). To generate an observer representation from this breakup we introduce observer actions for the photon and bradyon terms of (3.7a) and define a 'total action' as the sum of these observer partial actions and the remaining tachyon Stueckelberg–Jacobi partial actions. Thus, for example, if the segments between e_1 and $\bar{e}_1$ and between e_2 and $\bar{e}_2$ are timelike, the total action for (3.7a) is

$$A_T = A_\Omega[1\,\bar{1}] + A_\Omega[2\,\bar{2}] + A_{SJ}[\bar{1}\,\bar{2}] \tag{3.8}$$

An illustration is afforded in the example

$$\phi(x) = a^{-1}k[x^2 + (x\,.\,u)^2]^{-1/2} \tag{3.9}$$

where u is a constant unit timelike vector. Equation (3.6b) has solutions for $p_\lambda > 0$ (respectively < 0) when $k < 0$ (respectively > 0). The example of (3.8) corresponds to $p_\lambda > 0$ and the observer Lagrangian for the partial actions over the bradyon segments $e_1\bar{e}_1$ and $e_2\bar{e}_2$ is

$$L_\Omega = -m_b(1 - \dot{\mathbf{x}}^2)^{1/2}(1 + 2am_b^{-2}\phi(\mathbf{x}, t))^{1/2} \tag{3.10}$$

while for the tachyon part $\bar{e}_1\bar{e}_2$ we have the Stueckelberg–Jacobi Lagrangian

$$L_{SJ} = \pm m_b((x')^2)^{1/2}(-1 - 2am_b^{-2}\phi(x))^{1/2} \tag{3.11}$$

Notice that the observer representation *does not* distinguish the two possible directions of the four-dimensional Stueckelberg motion along the Minkowski trajectory. Solutions to the Stueckelberg equations for both signs of p_λ are given in Appendix A. Note in particular that equation (A.5) shows that the trajectories do not reverse invariantly in the time coordinate.

In the observer representation a free bradyon of mass m_b enters the region of the field. It accelerates to the speed of light when initial conditions are suitable and passes through the 'light barrier' into a tachyon phase, eventually slowing back down through the speed of light and emerging into the field-free asymptotic region with the mass with which it started. If $\bar{x}_1 - \bar{x}_2$ is spacelike there is a time interval over which initial and final bradyon particles are present essentially simultaneously.†

Physical laws corresponding to these features are straightforward statements about what happens to a (free) bradyon of mass m_b which is sent into the region of a Coulomb-like zero-degree Stueckelberg scalar field. Depending on initial conditions it may or may not go through the 'light barrier' into a tachyon phase. The case of tachyon asymptotic motion is similar, though here the laws govern the shape of the *entire* trajectory given $x^\mu(\lambda_1)$ and $x^\mu(\lambda_2)$.

A striking illustration of the importance of emphasizing the observer's role in the formulation of laws is provided in the example of the Stueckelberg 'time barrier' field,

$$\phi(x) = a^{-1} f(-x.u) \tag{3.12}$$

e.g. with

$$f(-x.u) = f(x^0) = -\frac{1}{2}\left(\frac{\kappa}{x^0}\right)^2 \tag{3.13}$$

in the frame with $u^0 = 1$, $u^1 = u^2 = u^3 = 0$. The solutions are worked out in Appendix B and examples of the possible trajectories are indicated in Fig. 1. As the most difficult case is where $p_\lambda > 0$ (asymptotic bradyon motion), we discuss only this.

Since initial conditions on the position and the velocity of a bradyon can be controlled, these can be specified by the observer at *both* ends of a trajectory of type A of Fig. 1. This would produce an over-determination of the curve, given the Stueckelberg or even just a set of Stueckelberg–Jacobi equations of motion. If only a single particle set of initial conditions are specified the observer finds that whenever he 'sends a particle into the region of a Stueckelberg "time barrier"', another particle always shows up miraculously to annihilate with it. No shield can bar the way to the intruder and we have a paradox.

† We use the term 'essentially simultaneous' to refer to a pair of spacelike separated events.

These arguments overlook an important difference between the time barrier field and that of the preceding example. There a part of the set of conditions (cause) leading to the scattering of the incident bradyon is the presence of the field asymptotically, namely, near $r = 0$. In the asymptotic limit the time barrier field vanishes over *all space*. *If* there are two particles present in the remote past and the initial conditions are suitable for the realization of curve A of Fig. 1, *then* their annihilation into a material event complex (called a tachyon), together with the accompanying emergence of the time barrier, occurs. Thus the time barrier field is to be regarded by an

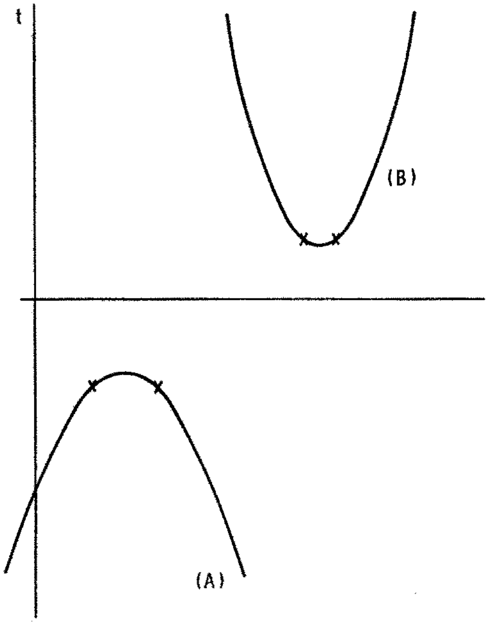

Figure 1.—Two examples of trajectories generated by the Stueckelberg scalar field 'time barrier'. The crosses mark the events $\bar{e}_i$ at which the world line tangent vectors are null.

observer, armed with his time sense, as a part of the *effect* consequent to the cause summarized in the conditions of the incident state. If initial conditions are unsuitable, e.g. if only one particle is present, then the time barrier never makes its appearance and we need a new Lagrangian.

If rules exist for preparing external fields, something should be said about what they are if they are to be objects of the theoretical framework at all. If the observer can assure the emergence of a Stueckelberg time-barrier field this must involve additional physical conditions in the initial state. Until these are given, no further conclusions can be drawn. (Example: an incident particle sees a gaseous bath of anti-particles, with rising density, as a time barrier. The timelike world lines of the anti-particles in the gas link the remote past (zero density) to $t=0$ (infinite density). Note that here the time-barrier field is a part of the cause of the annihilation rather than of its effect! Since its occurrence now has a causal link to the initial state this creates no difficulty.)

That the observer, in our problem, has no *a priori* knowledge of the existence of the Stueckelberg time-barrier field is a reflection of the primary experimental fact that he does *not live* (i.e. make observations, formulate laws, manipulate events) 'four-dimensionally'. He *lives* 'three-dimensionally', formulating laws in terms of four-dimensional invariants. He does not have the knowledge, at $t=-\infty$, of physical conditions near $t=0$, unless he takes suitable action guaranteed to produce, within experimental error, a known set of conditions. He lacks the space-time overview.

We proceed with the observer representation of the annihilation process based on (3.13), introducing the total action equation (3.8) with observer Lagrangian given by equations (B.12) and (B.13) in the frame where $u^0=1$, and $u^1=u^2=u^3=0$. The corresponding equations of motion give

$$\mathbf{p}_i = \frac{(m_b{}^2-\kappa^2/t^2)^{1/2}\mathbf{v}_i}{\sqrt{(1-\mathbf{v}_i)^2}} = \text{constant}, \qquad i=1,2 \tag{3.14}$$

so in this frame the annihilation of each of the particles occurs at constant 'momentum' by the 'fading' of the factor $M=(m_b{}^2-\kappa^2/t^2)^{1/2}$ with an attendant rise of v_i until the 'light barrier' is reached and M has fallen to zero. At this point a tachyon is 'produced' linking the two particles and the annihilation is complete with the (temporal) passage of the tachyon.

4. *Canonical Formulations and Lorentz-Invariant Tachyon Propagation*

Canonical formulations of Stueckelberg dynamics do not appear to contain anything of interest and would be beside the point in the present theory. Both the observer and Stueckelberg–Jacobi Lagrangians are reasonable starting points however.

For the free bradyon equation (2.37) gives the Hamiltonian

$$H=+(\mathbf{p}^2+m_b{}^2)^{1/2} \tag{4.1}$$

The Hamiltonian of the extended phase space vanishes identically and corresponding to this there is a constraint among the extended phase space canonical coordinates; in the present example it is

$$p_t+(\mathbf{p}^2+m_b{}^2)^{1/2}\approx 0 \tag{4.2}$$

This is a first-class constraint in the trivial sense and may be regarded as the starting point for canonical quantization, generating the Schrödinger equation of the quantized theory.

For free tachyons *there is no observer Lagrangian on which to base a canonical theory of the foregoing type*. Nevertheless it is possible to represent tachyons as possessing a Lorentz-invariant propagation character.

If we start from either of the Stueckelberg–Jacobi Lagrangians (2.29) we find the canonical constraint,

$$F(\epsilon^a) = [-g^{aa}(p_\mu p^\mu - p_a p^a + 2p_\lambda)]^{1/2} - g^{aa}\,\epsilon^a p_a \approx 0 \tag{4.3}$$

wherein the summation convention has been suspended for the index a, which is allowed to take any of the values from 0 to 3, and $\epsilon^a \equiv \epsilon(dx^a/d\lambda)$. For bradyons $2p_\lambda > 0$ and p_μ is a timelike vector on the constraint surface. But only for the choice $a = 0$ is the classification of the trajectories generated by the sign of ϵ^a in (4.3) Lorentz invariant. Correspondingly, bradyon coordinates may be regarded as representing a localized influence which propagates from spacelike hyperplanes, i.e. which evolves in time. For tachyons $2p_\lambda < 0$, p_μ is a spacelike vector on the constraint surface, and there is no choice of a for which the classification of the trajectories generated by the sign of ϵ^a in (4.3) is Lorentz invariant. Correspondingly, tachyon coordinates *may not* be regarded as representing a localized influence which propagates from either spacelike or timelike hyperplanes, in an invariant way.

These features are hidden when we replace the constraint (4.3) by

$$F = -g^{aa} F(\epsilon^a) F(-\epsilon^a) = p_\mu p^\mu + 2p_\lambda \approx 0 \tag{4.4}$$

The equations of motion in the extended phase space, with F as the Hamiltonian, then give extra solutions for each value of a, corresponding to propagation in which $dx^a/d\lambda < 0$. In the Klein–Gordon equation for bradyons these show up as the negative 'frequency' solutions and have to be disentangled from the solutions based on the Schrödinger equation by means of the causal propagator. The procedure is invariant for bradyons but the obvious generalizations to the tachyon case are not.

To solve the problem we have to find a canonical constraint which provides an invariant ϵ. There are two obvious ones, each corresponding to *spatial* propagation, in the variable $\zeta = (x^\mu x_\mu)^{1/2}$ in the one case and in the variable $r = (x^k x_k)^{1/2}$ in the other. We need only consider one of these, and the second will do.

If we introduce hypercylindrical coordinates relative to a given event e at the origin,

$$x^0 = t, \qquad x^1 = r\cos\theta\cos\phi$$
$$x^2 = r\cos\theta\sin\phi, \qquad x^3 = r\sin\theta \tag{4.5}$$

then we find the canonical constraint,

$$-\{m_t^2 + p_t^2 - r^{-2}[p_\theta^2 + csc^2\,\theta p_\phi^2]\}^{1/2} + \epsilon\frac{dr}{d\lambda}p_r \approx 0 \tag{4.6}$$

Thus the physically meaningful characterization of tachyon propagation, which is spatial rather than temporal, is (either inwards or outwards) through hypersurfaces of closed timelike cylinders or, what is topologically equivalent, to or from timelike lines.

There is a sense in which these separate characterizations of the propagation of particles and tachyons is unique. There are five classes of Lorentz-invariant hypersurfaces: the forward and backward light cones, the corresponding timelike pseudospheres and the spacelike pseudosphere. The light cones from a given event separate the other three classes of hypersurface centered at the same event. A world line which propagates spatially to a given event e passes through a concentric spacelike pseudosphere. The characterization of tachyon propagation in terms of timelike cylinders is provided by the one-to-one projection mapping of the pseudosphere onto the portion of a cylinder surrounding e which is caught between its intersections with the two branches of the double cone having vertex at e. An analogous construction is possible for particles.

Before turning to the photon we compare the 'squared constraints' for the two kinds of scalar field coupling given by equations (2.57a) and (3.1) For the former we find

$$p_\mu p^\mu + \epsilon(p_\lambda)\,[m - \epsilon(p_\lambda) f\Phi(x)]^2 \approx 0 \tag{4.7}$$

and for the latter,

$$p_\mu p^\mu + \epsilon(p_\lambda)\, m^2 + 2a\phi(x) \approx 0 \tag{4.8}$$

The first of these generalizes the minimal substitution rule to interaction with an external scalar field because it factors invariantly into forward- and backward-propagating particle and anti-particle solutions. The second of these lacks this property and gives the Klein–Gordon equation with an external Yukawa field; this observation makes the analysis of Section 3 more interesting. (Though the example of equation (3.9) corresponds to a zero mass 'pion' it does not appear that the inclusion of an exponential tail will affect any of its qualitative features.)

The free photon canonical formalism based on the observer Lagrangian of equation (2.44) involves a pair of second class constraints, $p_\omega \approx 0$ and $\omega - (\mathbf{p}^2)^{1/2} \approx 0$, which can be eliminated. The resulting total Hamiltonian is

$$H_T = +(\mathbf{p}^2)^{1/2} \tag{4.9}$$

The extended space Lagrangian based on the observer Lagrangian of equation (2.44) is

$$L_\Omega{}' = \tfrac{1}{2}\omega\left(\frac{\;}{t'} - t'\right) \tag{4.10}$$

which is a homogeneous function of the first degree in the generalized velocities. The corresponding Hamiltonian therefore vanishes identically and the concommitant canonical constraint is

$$p_t + (2\omega)^{-1}\,\mathbf{p}^2 + \tfrac{1}{2}\omega \approx 0 \tag{4.11}$$

which, after elimination of ω, p_ω by means of the second-class pair of constraints $p_\omega \approx 0$ and $\omega - (\mathbf{p}^2)^{1/2} \approx 0$, reduces to

$$p_t + (\mathbf{p}^2)^{1/2} \approx 0 \tag{4.12}$$

Thus canonical quantization generates the Schrödinger equation based on (4.9). The relations $p_\omega = 0$, $\omega - (\mathbf{p}^2)^{1/2} = 0$ then become equations defining p_ω and ω.

5. *Concluding Remarks*

The conventional causal framework regards the observer as, among other things, the interpreter of the regularities of the patterns of space-time events, with an event identified with a matter point at an instant. The present effort has been an attempt to make some of those ideas concrete in a simple context. It would have been natural to add another section to this paper dealing with the many-trajectory case where the Stueckelberg action integral is of a generalized Fokker type.† This would give a rather general kind of theory of the classical relativistic mechanical problem of interacting particles.

We have found the unique and invariant characterization of tachyon propagation and noted that this is *not causal*, i.e. not temporal, but is spatial. Tachyon 'production' can be controlled (caused) in time, however (Cawley, 1970). We have discussed canonical theories along with different approaches to canonical quantization, finding familiar classical field equations. We have not discussed the problem of the observer representation of the classical field theories. Finally, we have not considered discrete invariances.

Acknowledgement

The possible importance of a Jacobi's principle for Minkowski trajectories in relativistic particle mechanics was suggested to me a few years ago by Egon Marx, whom I wish to thank for many interesting and enlightening discussions.

Appendix A

Constant Scalar Field of Degree Zero

The scalar field example

$$\phi(x) = a^{-1} k[x^2 + (x.u)^2]^{-1/2} \equiv a^{-1} kr^{-1} \tag{A.1}$$

where u is a constant timelike unit vector, gives rise to Minkowski curves with tangent vectors whose signature can change. The Euler–Lagrange equations of the Stueckelberg action principle (2.3) with Stueckelberg Lagrangian (3.1) are

† Fokker, A. D. (1929). *Zeitschrift für Physik*, **58**, 386. If N Minkowski curves like the one in eq. (2.1) are given, then we could require that the coordinate function sets $x_n{}^\mu(\lambda)$, $n = 1$ to N, be derivable as solutions to a system of equations following from N action principles based on equations $\delta_n A_S = 0$, $n = 1$ to N, where the δ_n-variation of A_s is generated by infinitesimal fixed endpoint variations of the nth trajectory alone.

$$\frac{d^2 x^\mu}{d\lambda^2} = -a\phi'^\mu(x) = k[x^\mu + u^\mu(u.x)]\, r^{-3} \tag{A.2}$$

so choosing coordinates with $u^0 = 1$, $u^i = 0$, $i = 1, 2, 3$, we have

$$\frac{d^2 x^0}{d\lambda^2} = 0, \qquad \frac{d^2 x^i}{d\lambda^2} = kx^i r^{-3} \tag{A.3}$$

where $r = |x^i x_i|^{1/2} = |\mathbf{x}|$. From equation (3.2) we have

$$p_\lambda + \frac{1}{2}\left(\frac{d\mathbf{x}}{d\lambda}\right)^2 - \frac{1}{2}\left(\frac{dx^0}{d\lambda}\right)^2 = -kr^{-1} \tag{A.4}$$

From the first of equations (A.3),

$$\frac{dx^0}{d\lambda} = E = \text{constant} \tag{A.5}$$

The constancy of $dx^0/d\lambda$ simplifies the determination of the spatial part of the four-dimensional motion, for it allows the direct integration of equation (A.4). Introducing polar coordinates in the plane of the spatial motion we have

$$p_\lambda + \frac{1}{2}\left[\left(\frac{dr}{d\lambda}\right)^2 + j^2 r^{-2} - E^2\right] = -kr^{-1} \tag{A.6}$$

where the (positive) constant j is associated with the absence of θ-dependence in L_S; namely

$$j = r^2 \frac{d\theta}{d\lambda} \tag{A.7}$$

From equation (A.6) the distance of closest approach to $r = 0$ can be found as the smaller of the two solutions to

$$p_\lambda + \tfrac{1}{2}(j^2 r^{-2} - E^2) = -kr^{-1} \tag{A.8}$$

namely one of

$$r_\pm = (E^2 - 2p_\lambda)^{-1}\{k \pm [k^2 + j^2(E^2 - 2p_\lambda)]^{1/2}\} \tag{A.9}$$

We consider two cases simultaneously. In Case 1, $k < 0$ and we choose initial conditions so that $E^2 > 2p_\lambda > 0$ and $dx/d\lambda$ is timelike asymptotically. In Case 2, $k > 0$, $p_\lambda < 0$, and $dx/d\lambda$ is asymptotically spacelike. In both cases the distance of closest approach† is r_+ and at this point we have

$$\left(\frac{dx}{d\lambda}\right)^2 = j^2 r^{-2} - E^2 = \left\{-\frac{k}{j} + \left[\left(\frac{k}{j}\right)^2 + E^2 - 2p_\lambda\right]^{1/2}\right\}^2 - E^2 \tag{A.10}$$

The right side of (A.10) is positive in Case 1 if $jp_\lambda/E < -k$ and is negative in Case 2 if $-jp_\lambda/E < k$. In each of these examples the signature of the tangent vector is reversed as the curve crosses the surface of the Minkowski

† $d^2r/d\lambda^2 > 0$ holds at r_+ for the two examples considered.

cylinder‡ $r = r_0 = -k/p_\lambda$. Finally, we note that for $p_\lambda = 0$, $dx/d\lambda$ is null at infinity and can be either spacelike or timelike in close to the source, depending on the sign of k.

Appendix B

Uniform Time-Dependent Scalar Field of Degree Zero

Scalar field examples of the type

$$\phi(x) = a^{-1} f(-x.u) \equiv a^{-1} f(t), \qquad \text{with } \lim_{|t|\to\infty} f(t) = 0 \tag{B.1}$$

where u is a constant timelike unit vector, can generate Minkowski curves which reverse direction invariantly in the time coordinate. The Euler–Lagrange equations of the Stueckelberg action principle (2.3) with Lagrangian (3.1) are

$$\frac{d^2 x^\mu}{d\lambda^2} = -a\phi'^\mu(x) = u^\mu \dot{f}(t) \tag{B.2}$$

where the dot denotes differentiation with respect to the argument. With the choice of coordinates used in Appendix A we have

$$\frac{d^2 x^0}{d\lambda^2} = \frac{d^2 t}{d\lambda^2} = \dot{f}(t), \qquad \frac{d^2 x^i}{d\lambda^2} = 0, \qquad i = 1, 2, 3 \tag{B.3}$$

and from equation (3.2)

$$\tfrac{1}{2}E^2 - \frac{1}{2}\left(\frac{dt}{d\lambda}\right)^2 = -f(t) \tag{B.4}$$

where we have set

$$p_\lambda + \frac{1}{2}\left(\frac{d\mathbf{x}}{d\lambda}\right)^2 = \tfrac{1}{2}E^2 \tag{B.5a}$$

since, owing to $f(\infty) = 0$, the left side of (B.5a) is nonnegative, and where we have used

$$\mathbf{P} = \text{constant} = \frac{d\mathbf{x}}{d\lambda} \tag{B.5b}$$

which is the solution to the spatial part of the set, equations (B.3).

For definiteness we consider the example

$$f(t) = \frac{-\kappa^2}{2t^2} \tag{B.6}$$

for which equation (B.4) becomes

$$\left(\frac{dt}{d\lambda}\right)^2 = E^2 - \frac{\kappa^2}{t^2} \tag{B.7}$$

‡ One can verify that in the examples considered $r_0 > r_+$ when $|k| > j|p_\lambda|/E$.

For the example that $t \to -\infty$ at $\lambda \to -\infty$, the entire curve lies below the hyperplane

$$t = t_0 = \frac{-\kappa^2}{E} < 0 \tag{B.8}$$

and the time coordinates of the events e_i at which equations (3.6) hold are, for $p_\lambda > 0$,

$$\bar{t}_1 = \bar{t}_2 = \bar{t} = \frac{-\kappa}{m} < 0 \tag{B.9}$$

where $m = +(2p_\lambda)^{1/2}$. The curve is shown in Fig. 1 as curve A, and for suitable choice of origin it can be parametrized by

$$x^0 = t = -(t_0{}^2 + E^2\lambda^2)^{1/2}, \qquad \mathbf{x} = \mathbf{P}\lambda \tag{B.10}$$

Note that events e_1 and e_2 are essentially simultaneous.†

For $t < t_0$ there are two branches, whose trajectories are given by

$$\mathbf{x}_i(t) = \pm\frac{\mathbf{P}}{E}(t^2 - t_0{}^2)^{1/2}, \qquad t < t_0 < 0, \qquad i = 1, 2 \tag{B.11}$$

These represent two identical bradyon particles for $t < \bar{t}$ and the (formal) observer Lagrangian is

$$L_\Omega = L_{\Omega_1} + L_{\Omega_2} \tag{B.12}$$

where

$$L_{\Omega_i} = -m\left[1 - \left(\frac{\kappa}{mt}\right)^2\right]^{1/2}(1 - \dot{\mathbf{x}}_i{}^2)^{1/2}, \quad i = 1, 2 \tag{B.13}$$

For $\bar{t} < t < t_0$ the motion is faster than light.

The form (B.13) is actually not Lorentz invariant because we elected a specific frame to exhibit the features of the present example. This is remedied by replacing t with $(-x_i . u)$, where $x_1{}^0 = x_2{}^0 = t$ supplies the explicit t-dependence to L_{Ω_i}.

References

Cawley, R. G. (1970). *Conventional Spacetime Theory of Observation with Application to Hyperlight Phenomena*. NOL preprint.
Cawley, R. G. (1970). *Physical Review*, **D2**, 276.
Stueckelberg, E. C. G. (1941). *Helvetica physica acta*, **14**, 588.
Stueckelberg, E. C. G. (1942). *Helvetica physica acta*, **15**, 23.

† See footnote on p. 497.